Math 579 Fall 2013 Exam 8 Solutions

1. Solve the recurrence specified by $a_0 = 2, a_n = 3a_{n-1}$ $(n \geq 1)$.

   We have characteristic equation $r^n - 3r^{n-1} = 0$, which has single root $r = 3$. Hence the general solution is $a_n = A3^n$. We have $2 = a_0 = A3^0 = A$, so $A = 2$ and our solution is $a_n = 2 \cdot 3^n$.

2. Solve the recurrence specified by $a_0 = 2, a_n = 3a_{n-1} + 2$ $(n \geq 1)$.

   We first solve the homogeneous recurrence $a_n = 3a_{n-1}$, as in problem 1. Next we seek a single solution to the nonhomogeneous problem. Since 2 is a polynomial in n, and the general homogeneous solution isn't, we guess solution $a_n = k$ (constant polynomial). We solve $k = 3k + 2$ to get $k = -1$, and general nonhomogeneous solution $a_n = A3^n - 1$. Finally, we have $2 = a_0 = A3^0 - 1$, so $A = 3$ and our solution is $a_n = 3 \cdot 3^n - 1 = 3^{n+1} - 1$.

3. Solve the recurrence specified by $a_0 = 2, a_n = 3a_{n-1} + 3^n$ $(n \geq 1)$.

   We first solve the homogeneous recurrence $a_n = 3a_{n-1}$, as in problem 1. Next we seek a single solution to the nonhomogeneous problem. Since 3^n is an exponential with base 3, which is part of the homogeneous solution already, we instead guess $a_n = kn3^n$. We solve $kn3^n = 3(k(n-1)3^{n-1}) + 3^n = kn3^n - k3^n + 3^n$, which has solution $k = 1$. Hence the general nonhomogeneous solution is $a_n = A3^n + n3^n$. Finally, we have $2 = a_0 = A3^0 - 0 \cdot 3^0$, so $A = 2$ and our solution is $a_n = 2 \cdot 3^n + n3^n$.

4. Let a_n be the number of ways to fill a $2 \times n$ chessboard using white 1×1 squares, red 2×2 squares, and blue 2×2 squares. Find a recurrence and a closed form for a_n.

   Looking at the initial part of the chessboard, the first column could be filled with two 1×1 squares, or the first two columns could be filled with either a red or blue 2×2 square. Hence $a_n = a_{n-1} + 2a_{n-2}$. This has characteristic equation $r^2 - r - 2 = 0$, which has two roots $r = 2, -1$. Thus the general solution is $a_n = A2^n + B(-1)^n$. Our initial conditions are $a_0 = a_1 = 1$, which gives us equations $A + B = 1, 2A - B = 1$. This has solutions $A = \frac{2}{3}, B = \frac{1}{3}$, so the final answer is $a_n = \frac{1}{3}(2^{n+1} + (-1)^n)$.

5. Solve the recurrence specified by $a_0 = 2, a_1 = 3, a_2 = 8$, and (for $n \geq 3$), $a_n = -a_{n-1} + a_{n-2} + a_{n-3} + 8$.

   We first consider the homogeneous recurrence, leaving off the final 8. The characteristic equation is $0 = r^3 + r^2 - r - 1 = (r+1)^2(r-1)$. Hence the general solution is $a_n = A(-1)^n + Bn(-1)^n + C(1)^n = (A + nB)(-1)^n + C$. Turning to the nonhomogeneous problem, the 8 is a polynomial in n; however we already have zeroth-degree polynomials represented in the homogeneous problem. Hence we guess second-degree polynomial $a_n = kn$. We have $kn = -k(n-1) + k(n-2) + k(n-3) + 8$. Like magic, all the terms with n drop out, and we're left with $0 = k - 2k - 3k + 8$, which has solution $k = 2$. Hence the general solution is $a_n = (A + nB)(-1)^n + C + 2n$. We have $2 = a_0 = A(-1)^0 + C$, $3 = a_1 = (A + B)(-1)^1 + C + 2$, $8 = (A + 2B)(-1)^2 + C + 4$, which has solution $A = 0, B = 1, C = 2$. Hence the final answer is $a_n = n(-1)^n + 2 + 2n$.